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Bipartite Posets of Finite Prinjective Type

Hans-Joachim von Höhne

Institut für Mathematik II, Freie Universität Berlin, 14 195 Berlin, Germany

and

Faculty of Mathematics and Informatics, Nicholas Copernicus University, 87-100 Toruń, Poland

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One of the main results of this paper is Theorem 1.2, which contains a characterization of finite bipartite posets $I = I' \cup I''$ for which the category $\operatorname{prin}(kI)$ of prinjective modules over the incidence k-algebra kI of I is of finite representation type, where k is a field. In particular, it is shown that for any such poset I, the Auslander–Reiten quiver of the category $\operatorname{prin}(kI)$ has no oriented cycle, and there is a bijection between the isomorphism classes of indecomposable objects in $\operatorname{prin}(kI)$ and the positive roots of the quadratic form q^I associated with I. An existence of a preprojective component in $\operatorname{prin}(kI)$ is proved for faithful posets I which are $\tilde{\mathbb{A}}$-free. © 1998 Academic Press

1. INTRODUCTION

(1.1) Let k be a field, let $(I, \preceq)$ be a finite partially ordered set (abbreviated-poset), and denote by kI the incidence algebra of I over k, that is, the k-vector space with basis $\{e_{ij} \mid i, j \in I, i \preceq j\}$ and multiplication given by $e_{ij}e_{kl} = \delta_{j\kappa}e_{il}$, where $\delta_{j\kappa}$ is the Kronecker delta. In this paper we are concerned with modules over kI, where I is *bipartite*, that is, I is equipped with a partition

$$I = I' \cup I''$$

such that I' and I'' are subposets of I and $j \not\preceq i$ for all $i \in I'$, $j \in I''$. For such a bipartite poset $I = I' \cup I''$, the incidence algebra kI is isomorphic

*E-mail address: simson@mat.uni.torun.pl. Fax: (56)28979.

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to the triangular matrix algebra

$$\begin{pmatrix} kI' & M \\ 0 & kI'' \end{pmatrix}$$

where M is the kI'-kI''-bimodule with k-basis $\{e_{ij} \mid i \in I', j \in I'', i \prec j\}$. It is easy to check that the algebra kI is of finite global dimension.

A right module X over kI will be identified with the triple

$$X = (X', X'', \varphi)$$

where $X' = \oplus_{i \in I'} Xe_{ii}$ is viewed as a kI'-module, $X'' = \oplus_{j \in I''} Xe_{jj}$ is viewed as a kI''-module, and

$$\varphi : X' \otimes_{kI'} M \to X''$$

is the kI''-module homomorphism defined by the multiplication $\cdot : X \times kI \to X$.

Following [18] and [28], the kI-module $X = (X', X'', \varphi)$ is called *prinjective* if the kI'-module X' is projective and the kI''-module X'' is injective. We denote by $\operatorname{prin}(kI)$ the category of finite-dimensional prinjective kI-modules.

It follows from [18] and [28] that the category $\operatorname{prin}(kI)$ is closed under extensions in $\operatorname{mod}(kI)$ and has almost split sequences. We denote by $\Gamma(\operatorname{prin}(kI))$ the Auslander−Reiten quiver of the category $\operatorname{prin}(kI)$ (see [4], [18], and [23, 11.2]).

The bipartite poset $I = I' \cup I''$ is said to be of *finite prinjective type* if the category $\operatorname{prin}(kI)$ has only finitely many indecomposable objects up to isomorphism.

(1.2) The main result of this paper is the following.

THEOREM. *For a bipartite poset $I = I' \cup I''$, the following conditions are equivalent*:

(i) *The bipartite poset I is of finite prinjective type*.

(ii) *The Tits quadratic form $q^I : \mathbb{Z}^I \to \mathbb{Z}$ of $I = I' \cup I''$ defined by the formula*

$$q^I(x) = \sum_{i \in I} x_i^2 + \sum_{\substack{i \prec j \\ i, j \in I'}} x_i x_j + \sum_{\substack{i \prec j \\ i, j \in I''}} x_i x_j - \sum_{\substack{i \prec j \\ i \in I', j \in I''}} x_i x_j$$

is weakly positive, that is, $q^I(x) > 0$ for all nonzero vectors $x \in \mathbb{N}^I$, where $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of nonnegative integers.

(iii) *The bipartite poset I does not contain as a bipartite subposet any of the 41 bipartite posets $I_1, \dots, I_{41}$ presented in Figure 1 or its bipartite dual posets*.

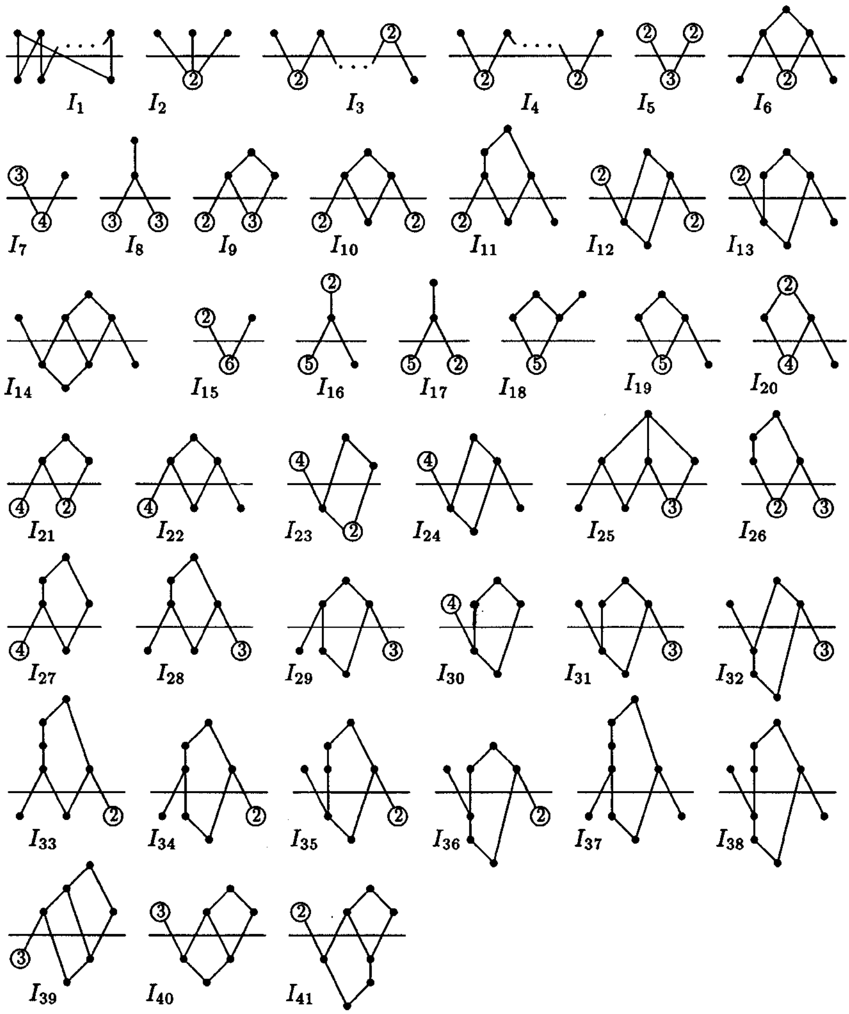


FIG. 1. Critical bipartite posets $I_1, \ldots, I_{41}$ (up to duality).

(iv) *There exists a preprojective component* $\tilde{P}(I)$ (*see* [4], [18], [23, *p.* 200]) *of the Auslander–Reiten quiver* $\Gamma(\mathrm{prin}(kI))$ *and* $\Gamma(\mathrm{prin}(kI)) = \tilde{\mathscr{P}}(I)$.

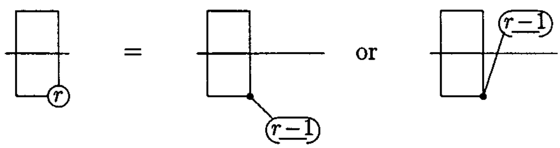
(v) *The set*

$$\mathscr{R}^{+}_{q^I} = \{v \in \mathbb{N}^I;\ q^I(v) = 1\}$$

of positive roots of the quadratic form q^I *is finite, and the map* $X \mapsto \mathbf{cdn}\, X$ (*see* (2.1)) *defines a bijection* $\mathbf{cdn} : \Gamma(\mathrm{prin}(kI)) \to \mathscr{R}^{+}_{q^I}$.

Here a *bipartite subposet* of $I = I' \cup I''$ is a subposet $J \subseteq I$ equipped with the bipartition $J = (I' \cap J) \cup (I'' \cap J)$. The *dual* J^* of a bipartite poset $J = J' \cup J''$ is the opposite poset J^{op} together with the bipartition $J^* = (J'')^{\mathrm{op}} \cup (J')^{\mathrm{op}}$.

CONVENTION. *The following convention will be used in the diagrams* $I_1, \dots, I_{41}$ *above. Each vertex with label* (r) *admits* 2^{r-1} *possible extensions, recursively defined as follows*:



where the process stops at vertices with label (1) (*see the end of Section* 4 *for more details*). *The bipartition of* I_j *is marked by a horizontal line.*

As a consequence, we obtain the following.

COROLLARY. *If the bipartite poset* $I = I' \cup I''$ *is of finite prinjective type, then the Auslander–Reiten quiver* $\Gamma(\mathrm{prin}(kI))$ *of the category* $\mathrm{prin}(kI)$ *has no oriented cycle*, $\mathrm{End}(X) \simeq k$ *and* $\mathrm{Ext}^1_{kI}(X, X) = 0$ *for all indecomposable modules* X *in* $\mathrm{prin}(kI)$.

The theorem generalizes the results of [15] and [22, Theorem 3.1] (see also [24] and Chapter 16 of [23]). We hope that an extension of Theorem 1.2 in the form of [16, Theorem 2.1] is valid for bipartite posets.

To prove the equivalence of (i) and (ii) in Theorem 1.2, we introduce the notion of ρ-separation of the bipartite algebra kI (see Definition 3.2)). It turns out that this is a suitable prinjective module analogue of the separation property introduced by Bautista and Larrion in [6]. The proof of the equivalence of (ii) and (iii) is based on the main results of [11] and [29].

(1.3) The paper is organized as follows. In Section 2 we recall from [18] some facts about the category $\mathrm{prin}(kI)$. In Lemma 2.5 we describe the decomposition of almost split morphisms ending in any indecomposable prin-projective module. In Section 3 we show that the Auslander–Reiten quiver $\Gamma(\mathrm{prin}(kI))$ of the category $\mathrm{prin}(kI)$ has a preprojective component if the bipartite poset I is faithful and $\tilde{\mathbb{A}}$-free, by applying the fact that kI has the ρ-separation property in this case. Using a result of [18], this gives the equivalence of (i) and (ii) in Theorem 1.2. The equivalence of (i), (ii), (iv), and (v) in Theorem 1.2 is proved in (3.6). In Section 4 we show how the equivalence of (ii) and (iii) follows from [11] and [29].

In Section 5 an interpretation of prinjective kI-modules is given in terms of a matrix problem language (see [10], [23]). It is also shown that $I = I' \cup I''$ is of finite prinjective type if and only if for every coordinate vector $(v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$ the irreducible matrix k-variety $\mathbf{Mat}^I_{(v',v'')}$ (associated with (v', v'') and I) has only finitely many $\mathbf{G}^I_{(v',v'')}$-orbits with respect to an action (defined in (5.3))

$$\bullet : \mathbf{G}^I_{(v',v'')} \times \mathbf{Mat}^I_{(v',v'')} \to \mathbf{Mat}^I_{(v',v'')}$$

of an algebraic group $\mathbf{G}^I_{(v',v'')}$ or, equivalently, if and only if $\dim \mathbf{G}^I_{(v',v'')} > \dim \mathbf{Mat}^I_{(v',v'')}$ for any $(v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$, where dim means the k-variety dimension. Moreover, we show that the category $\operatorname{prin}(kI)$ is of tame representation type (see [23, Section 14.4]) if and only if for every coordinate vector $(v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$ there exists a constructible subset $\mathscr{C}(v', v'')$ of the subvariety $\mathbf{ind}\,\mathbf{Mat}^I_{(v',v'')}$ of $\mathbf{Mat}^I_{(v',v'')}$, defined by the indecomposable $\mathbf{G}^I_{(v',v'')}$-orbits such that $G^I_{(v',v'')} \bullet \mathscr{C}(v', v'') = \mathbf{ind}\,\mathbf{Mat}^I_{(v',v'')}$ and $\dim \mathscr{C}(v', v'') \leq 1$.

Throughout this paper we use the terminology and notation introduced in [18] and [23]. The reader is referred to [1], [2], [3], [10], [19], [22], [23], [25], and [26] for a motivation and application of representations of posets.

2. SINK MAPS ENDING IN PRIN-PROJECTIVE MODULES

(2.1) Throughout we assume that $I = I' \cup I''$ is a bipartite poset. As mentioned in the Introduction, the incidence algebra kI of I is isomorphic to the triangular matrix algebra

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix},$$

where $A = kI'$, $B = kI''$, and M is an A-B-bimodule. We denote by $\operatorname{mod}(R)$ the category of finite-dimensional right R-modules and describe R-modules by triples (X', X'', φ), where X' is in $\operatorname{mod}(A)$, X'' is in $\operatorname{mod}(B)$, and $\varphi : X' \otimes_A M \to X''$ is a B-homomorphism. A morphism $g : (X', X'', \varphi) \to (Y', Y'', \psi)$ is given by a pair of maps (g', g''), where $g' : X' \to Y'$ is an A-homomorphism, $g'' : X'' \to Y''$ is a B-homomorphism, and $\psi(g' \otimes \mathrm{id}_M) = g''\varphi$.

Following [18] and [28], the kI-module $X = (X', X'', \varphi)$ is called *prinjective* if the kI'-module X' is projective and the kI''-module X'' is injective. We denote by $\operatorname{prin}(kI)$ the full subcategory of $\operatorname{mod}(kI)$ formed by all prinjective modules.

It follows from [18], Proposition 2.4, Theorem 3.4] and [21, Lemma 5.11] that $\operatorname{prin}(kI)$ is a Krull–Schmidt category, is closed under extensions in $\operatorname{mod}(kI)$, and has almost split sequences. Moreover, the category $\operatorname{prin}(kI)$ has enough relative projective objects and enough relative injective objects, and $\operatorname{Ext}^1_{kI}(X, Y) = 0$ for any pair X, Y of modules $\operatorname{prin}(kI)$.

For each $i \in I$, we set $e_i = e_{ii}$ and choose a simple R-module S_i such that $S_i e_i = S_i$. The *coordinate vector* of a prinjective module X in $\operatorname{prin}(R)$ is the integral vector $\mathbf{cdn}\, X \in \mathbb{N}^I$ defined by the formula (see [20], [18])

$$(\mathbf{cdn}\, X)_i = \begin{cases} \dim_k \operatorname{Hom}_R(X, S_i) & \text{if } i \in I', \\ \dim_k \operatorname{Hom}_R(S_i, X) & \text{if } i \in I''. \end{cases}$$

Following [23] and [24], the bipartite subposet

$$\mathbf{csupp}\, X = \{ i \in I \mid (\mathbf{cdn}\, X)_i \neq 0 \}$$

of I is called the *coordinate support* of X.

Let $\mathbf{K}_0(\operatorname{prin}(kI))$ be the Grothendieck group of the category $\operatorname{prin}(kI)$. It is easy to see that the map $X \mapsto \mathbf{cdn}\, X$ induces a group isomorphism $\mathbf{K}_0(\operatorname{prin}(kI) \cong \mathbb{Z}^{I'} \times \mathbb{Z}^{I''}$ (see [18], [27]).

(2.2) If J is a bipartite subposet of I, we have an isomorphism of k-algebras

$$kJ \simeq \begin{pmatrix} eAe & eMf \\ 0 & fBf \end{pmatrix},$$

where $e = \sum_{j \in J'} e_j$ and $f = \sum_{j \in J''} e_j$. We denote by

$$\mathscr{T}_J^I : \operatorname{prin}(kJ) \to \operatorname{prin}(R)$$

the induction functor, which maps a prinjective kJ-module $Y = (Y', Y'', \varphi)$ onto

$$\mathscr{T}_J^I Y = \left(Y' \otimes_{eAe} eA, \operatorname{Hom}_{fBf}(Bf, Y''), \hat{\varphi} \right),$$

where $\hat{\varphi}(y \otimes a \otimes m)(b) = \varphi(y \otimes amb)$ for all $y \in Y', a \in eA, m \in M$, and $b \in Bf$.

The following lemma is easily verified (compare with [21, 2.24]).

LEMMA. (*a*) *For any module Y in* $\operatorname{prin}(kJ)$, *the following equality holds*:

$$\left(\mathbf{cdn}\, \mathscr{T}_J^I Y\right)_j = \begin{cases} (\mathbf{cdn}\, Y)_j & \text{if } j \in J, \\ 0 & \text{otherwise}. \end{cases}$$

(*b*) *The functor* $\mathscr{T}_J^I$ *is full and faithful, and induces an equivalence between* $\operatorname{prin}(kJ)$ *and a full subcategory of* $\operatorname{prin}(R)$ *formed by the modules* X *with* $\mathbf{csupp}\, X \subseteq J$.

(*c*) *The functor* $\mathscr{T}_J^I$ *maps short exact sequences into short exact sequences.*

We shall often identify $\operatorname{prin}(kJ)$ *with the image of* $\mathscr{T}_J^I$.

(2.3) *With any R-module* $X = (X', X'', \varphi)$ *we associate two R-modules,*

$$X^{\wedge} = \big(P_A(X', X''), \varphi(\varepsilon \otimes \mathrm{id}_M)\big) \qquad \text{and} \qquad X^{\vee} = \big(X', E_B(X''), \eta\varphi\big),$$

where $\varepsilon : P_A(X') \to X'$ *is a projective cover in* $\operatorname{mod}(A)$ *and* $\eta : X'' \to E_B(X'')$ *is an injective envelope in* $\operatorname{mod}(B)$. *Clearly, there is an isomorphism* $X^{\wedge\vee} \simeq X^{\vee\wedge}$, *and the modules* $X^{\wedge\vee}, X^{\vee\wedge}$ *are in* $\operatorname{prin}(R)$.

Remark. The map $X \mapsto X^{\vee\wedge}$ extends to a functor

$$\Xi : \operatorname{mod}(R) \to \overline{\operatorname{prin}}(R),$$

where $\overline{\operatorname{prin}}(R)$ denotes the residue category of $\operatorname{prin}(R)$ modulo the ideal of morphisms which factor through a module of the form $(Z', Z'', 0)$ in $\operatorname{prin}(R)$: a morphism $g = (g', g'') : X \to Y$ is mapped under Ξ onto the residue class $\overline{h}$, where $h = (h', h'') : X^{\vee\wedge} \to Y^{\vee\wedge}$ is such that $\varepsilon_Y h' = g'\varepsilon_X$ and $h''\eta_X = \eta_Y g''$.

Let $\operatorname{adj}(R)$ denote the full subcategory of $\operatorname{mod}(R)$ formed by the *adjusted* R-modules (see [21], [18]), that is, the R-modules $X = (X', X'', \varphi)$ such that φ is surjective and the adjoint map $\overline{\varphi} : X' \to \operatorname{Hom}_B(M, X'')$ of φ is injective or, equivalently, $(\operatorname{soc} X)A = 0 = (\operatorname{top} X)B$ (see [21]). The restriction of Ξ to $\operatorname{adj}(R)$ yields an equivalence

$$\operatorname{adj}(R) \approx \overline{\operatorname{prin}}(R),$$

and the quasi-inverse of Ξ is induced by the adjustment functor $\Theta : \operatorname{prin}(R) \to \operatorname{adj}(R)$ defined by the formula $\Theta(X', X'', \varphi) = (\operatorname{Im}\overline{\varphi}, \operatorname{Im}\varphi, \tilde{\varphi})$, where $\tilde{\varphi}(\overline{\varphi}(x) \otimes m) = \varphi(x \otimes m)$ for all $x \in X', m \in M$ [18, 3.3].

(2.4) According to [18], the category $\operatorname{prin}(R)$ has enough relative projective modules (called *prin-projective* modules), and any indecomposable prin-projective module is isomorphic to one of the modules $B_i, i \in I$, defined as follows:

$$B_i = \begin{cases} (e_i R)^{\vee} & \text{if } i \in I', \\ (S_i)^{\vee} & \text{if } i \in I''. \end{cases}$$

Furthermore, $\operatorname{prin}(R)$ has enough relative injective modules (called *prin-injective* modules), and any indecomposable prin-injective module is isomorphic to one of the following modules:

$$U_i = \begin{cases} (S_i)^{\wedge} & \text{if } i \in I', \\ (DRe_i)^{\wedge} & \text{if } i \in I''. \end{cases}$$

Here and in the following, $D = \operatorname{Hom}_k(-, k)$ denotes the k-duality functor. Moreover, $\operatorname{prin}(R)$ has almost split sequences. In particular, for any $i \in I$, there exist a minimal right almost split morphism ending at B_i

$$\rho_i : \rho(B_i) \to B_i,$$

and a minimal left almost split morphism starting from U_i,

$$\lambda_i : U_i \to \lambda(U_i).$$

The modules $\rho(B_i)$ and $\lambda(U_i)$ are uniquely determined up to isomorphism, and the modules $\rho(B_i)$ have the following description.

LEMMA. (*a*) *For every* $i \in I'$, *we have* $\rho(B_i) \simeq (\operatorname{rad} e_i R)^{\vee\wedge}$.

(*b*) *For every* $i \in I''$, *we have* $\rho(B_i) \simeq (\tau_R S_i)^{\vee\wedge}$, *where* τ_R *denotes the Auslander–Reiten translation in* $\operatorname{mod}(R)$ (*see* [4], [5], [23]).

Proof. The statement (a) follows from [18, 2.5].

(b) Fix an index $i \in I''$ and let $P \xrightarrow{\beta} e_i B$ be a minimal projective presentation of S_i in $\operatorname{mod}(B)$. The Nakayama functor

$$\mathcal{N}_B = D\operatorname{Hom}_B(-, B) : \operatorname{mod}(B) \to \operatorname{mod}(B)$$

maps β into a sink map in the category $\operatorname{inj}(B)$ of finite-dimensional injective B-modules, and according to [18, 2.5] we have

$$\rho(B_i) \simeq \mathcal{K}(\operatorname{Ker}\mathcal{N}_B(\beta))^{\vee\wedge},$$

where

$$\mathcal{K} : \operatorname{mod}(B) \to \operatorname{mod}(R)$$

$X'' \mapsto (\operatorname{Hom}_B(M, X''), X'', \nu)$, denotes the "coinduction functor" (here ν is the evaluation map). On the other hand, if we identify $\operatorname{mod}(B)$ with a full subcategory of $\operatorname{mod}(R)$ via the algebra surjection $R \to B$ and denote by $\operatorname{proj}(B)$ the category of finite-dimensional projective B-modules, then the Nakayama functor $\mathcal{N}_R : \operatorname{mod}(R) \to \operatorname{mod}(R)$ satisfies

$$\mathcal{N}_R|_{\operatorname{proj}(B)} \simeq \mathcal{K}\mathcal{N}_B|_{\operatorname{proj}(B)}.$$

Since $\mathscr{K}$ is left exact and β is also a minimal projective presentation of S_i in $\operatorname{mod}(R)$, we obtain $\mathscr{K}(\operatorname{Ker}\mathscr{N}_B(\beta)) \simeq \operatorname{Ker}\mathscr{K}\mathscr{N}_B(\beta) \simeq \operatorname{Ker}\mathscr{N}_R(\beta) \simeq \tau_R S_i$. ∎

(2.5) So far we have not really used the partial order relation $\preceq$ on I. It becomes more relevant now when we examine how the modules $\rho(B_i)$ decompose into indecomposables.

For this purpose we consider a given bipartite subposet J of I, and denote by $\sim$ the smallest equivalence relation on J which satisfies $i \sim j$ whenever $i \in J', j \in J''$, and $i \prec j$. We call the equivalence classes with respect to $\sim$ the *connected components* of the bipartite poset J and say that J is *connected* if any two elements of J are equivalent. For instance, if X is an indecomposable module in $\operatorname{prin}(R)$, then $\mathbf{csupp}\, X$ is a connected bipartite subposet of I.

Now let J be a bipartite subposet of I such that J' and J'' are *antichains*, that is, $i \not\prec j$ for all $i, j \in J'$ of J'', respectively. We associate with J the prinjective R-module

$$T(J) = \left(\bigoplus_{i \in J'} e_i A, \bigoplus_{j \in J''} DBe_j, \varphi \right),$$

where $\varphi(e_i \otimes e_{ij}) = e_j^*$ (the dual-basis vector to e_j) for all $i \in J', j \in J''$ with $i \prec j$. Clearly, $(\mathbf{cdn}\, T(J))_i = 1$ if $i \in J$, and $(\mathbf{cdn}\, T(J))_i = 0$ otherwise. Moreover, if the connected components of J are denoted by $J_1, \dots, J_r$, then

$$T(J) = \bigoplus_{i=1}^{r} T(J_i)$$

is a decomposition into indecomposable R-modules.

Given a subset $L \subseteq I$ and an element $i \in I$, we set

$$L_{\diamond i} = \{ l \in L \mid l \diamond i \},$$

where $\diamond$ is any of the symbols $\prec, \preceq, \succ, \succeq, \not\prec$ etc..

LEMMA. (*a*) *If* $i \in I'$, *then* $\rho(B_i) \simeq T(J_1) \oplus \cdots \oplus T(J_r)$, *where* $J_1, \dots, J_r$ *are the connected components of the bipartite subposet* $J = \min(I'_{\succ i}) \cup \max(I''_{\succ i})$ *of* I.

(*b*) *If* $i \in I'' \setminus \max I''$ *and* $I'_{\prec i} \neq \emptyset$, *then* $\rho(B_i)$ *is indecomposable and*

$$(\mathbf{cdn}\,\rho(B_i))_j = \begin{cases} 1 & \text{if } j \in \hat\imath, \\ |\hat\imath| - 1 & \text{if } j \in (\min I')_{\prec i}, \\ |\hat\jmath| & \text{if } j \in (\min I')_{\not\prec i}, \\ 1 & \text{if } j \in \min(I'_{\not\prec i}) \setminus \min I',\ \hat\jmath = \hat\imath, \\ 0 & \text{otherwise}; \end{cases}$$

here $\hat\jmath = (\min(I''_{\succ i}))_{\succeq j}$; *in particular*, $\hat\imath = \min(I''_{\succ i})$.

(*c*) *If* $i \in I'' \setminus \max I''$ *and* $I'_{\prec i} = \emptyset$, *then* $\rho(B_i) \simeq \bigoplus_{j \in \min(I'_{\succ i})} U_j$.

(*d*) *If* $i \in \max I''$, *then* $\rho(B_i) = 0$.

Proof. (a) follows from Lemma 2.4 and the isomorphism $(\operatorname{rad} e_i R)^{\vee\wedge} \simeq T(J)$.

(b) In the following we denote by Q the quiver (= Hasse diagram) of I and describe R-modules by k-linear representations of Q.

Since $\rho(B_i) \simeq (\tau_R S_i)^{\vee\wedge}$ (Lemma 2.4), we examine first the top and the socle of $\tau_R S_i$. Consider the k-linear representation W of the quiver Q defined by $W(p) = k^{\hat p}$ for each vertex p of Q and where $W(\alpha): k^{\hat p} \to k^{\hat q}$ is the canonical projection for each arrow $p \xrightarrow{\alpha} q$ of Q (note that $\hat p \supseteq \hat q$ if $p \prec q$). The representation W is injective, and its socle is isomorphic to $\oplus_{j \in \hat\imath} S_j$. So $\tau_R S_i$ is given by the subrepresentation V of W defined by $V(p) = W(p)$ if $p \not\preceq i$, and $V(p) = H := \{x \in k^{\hat\imath} \mid \sum_j x_j = 0\}$ otherwise (note that $\hat p = \hat\imath$ if $p \preceq i$). Clearly, $\operatorname{soc} V = \operatorname{soc} W$, $W(\alpha)$ is surjective for each arrow $p \xrightarrow{\alpha} q$ of Q, and $V(\alpha)$ is so unless $p \preceq i$, $q \not\preceq i$, and $\hat q = \hat\imath$. In this case, $\operatorname{Im} V(\alpha) = H$ has codimension 1 in $k^{\hat q}$ and is supplemented by any canonical basis vector of $k^{\hat q}$. The above formula for $\mathbf{cdn}\,\rho(B_i)$ follows easily.

It remains to show that $(\tau_R S_i)^{\vee\wedge}$ is indecomposable. For this purpose we choose an associated representation F of Q and examine its endomorphism ring. Of course, it is enough to consider F only on the vertices of the coordinate support. We have

$$F(p) = \begin{cases} V(p) = k^{\{p\}} & \text{if } p \in \hat\imath, \\ V(p) = H^{\{p\}} & \text{if } p \in (\min I')_{\prec i}, \\ V(p) = k^{\hat p} & \text{if } p \in (\min I')_{\not\prec i}, \\ k^{\{\kappa\}} \oplus H^{\check p} & \text{if } p \in L; \end{cases}$$

here $L = \{j \in \min(I'_{\nprec}) \mid I'_{\prec j} \neq \emptyset, \hat{\jmath} = \hat{\imath}\}$, $\check{p} = ((\min I')_{\prec i})_{\prec p}$, and $\kappa \in \hat{\imath}$ denotes some fixed element. For any two elements p, q of $\mathbf{csupp} F$ which satisfy $p \prec q$, the k-linear map $F_{qp} : F(p) \to F(q)$ is independent of the chosen path from p to q in Q and has the following form:

$$F_{qp} = \begin{cases} V_{qp} & \text{if } p \in \min I', q \in \hat{\imath}, \\ \left[\, \delta_{q\kappa} \Pi_q \right] : k^{\{\kappa\}} \oplus H^{\check{p}} \to k^{\{q\}} & \text{if } p \in L, q \in \hat{\imath}, \\ \left[0 \eta_p \right]^{\mathsf{T}} : H^{\{p\}} \to k^{\{\kappa\}} \oplus H^{\check{q}} & \text{if } p \in (\min I')_{\prec i}, q \in L. \end{cases}$$

Here $\eta_p : H^{\{p\}} \to H^{\check{q}}$ is the pth canonical injection, and Π_q is the map $\sigma(\pi_q \mid_H)^{\check{p}}$, where $\pi_q : k^{\hat{\imath}} \to k^{\{q\}}$ is the qth canonical projection and $\sigma : (k^{\{q\}})^{\check{p}} \to k^{\{q\}}$ is the sum map. Now let $(\varphi_j)_{j \in I}$ be an endomorphism of F. The map $\phi = \oplus_{j \in \hat{\imath}} \varphi_j \in \mathrm{End}_k(\oplus_j F(j)) = \mathrm{End}_k(k^{\hat{\imath}})$ is a dilatation (with the canonical basis as eigenbasis), which leaves the "skew" hyperplane $H = F(p)$ invariant for any $p \in (\min I')_{\prec i} (\neq \emptyset!)$. This implies that $\phi = \lambda \mathrm{id}_{k^{\hat{\imath}}}$ for some $\lambda \in k$; therefore $\varphi_j = \lambda$ for all $j \in \hat{\imath}$, and $\varphi_p = \phi \mid_{F(p)} = \lambda \mathrm{id}_{F(p)}$ for all $p \in \min I'$. Finally, if $q \in L$, then $\varphi_q \mid_{H^{\check{q}}} = \oplus_{p \in \check{q}} \varphi_p = \lambda \mathrm{id}_{H^{\check{q}}}$, and $F_{\kappa q} \varphi_q = \varphi_\kappa F_{\kappa q}$ implies that $\varphi_q(e_\kappa) \in \lambda e_\kappa + H^{\check{q}}$. Thus F has a local endomorphism ring.

(c) As in (b), we have $(\mathbf{cdn}\, \rho(B_i))_j = 1$ if $j \in \hat{\imath}, |\hat{\jmath}|$ if $j \in \min I'$, and 0 otherwise. Comparing coordinate vectors, it therefore is enough to show that U_j is a direct summand of $\rho(B_i)$ for every $j \in \min(I''_{\succ i})$, or equivalently, that $(S_i)^\vee$ is a direct summand of $\lambda(U_j)$. Indeed, this follows from the dual of (a) and the assumption $I'_{\prec i} = \emptyset$, noting that $i \in \max(I''_{\prec j})$ if $j \in \min(I''_{\succ i})$.

(d) Since S_i is a simple projective R-module, we have $\tau_R S_i = 0$. ∎

3. EXISTENCE OF A PREPROJECTIVE COMPONENT

In this section we prove the equivalence of the statements (i), (ii), (iv), and (v) in our main theorem (see Section 1). We reduce the proof fo the case where the bipartite poset I is faithful and $\tilde{\mathbb{A}}$-free, and show that in this case the Auslander–Reiten quiver $\Gamma(\mathrm{prin}(kI))$ of the category $\mathrm{prin}(kI)$ (see [4], [23, Section 11.1]) has a preprojective component. We then can apply a result of [18, Proposition 4.13]. The details are given in (3.6).

(3.1) Given a bipartite poset $I = I' \cup I''$, we call the bipartite subposet

$$J = \left\{ i \in I' \mid I''_{\succ i} \neq \emptyset \right\} \cup \left\{ j \in I'' \mid I'_{\prec j} \neq \emptyset \right\}$$

the *faithful part* of I and say that I is *faithful* if $J = I$. By Lemma 2.2 (b), the induction functor $\mathcal{I}_J^I$ induces an equivalence between $\operatorname{prin}(kJ)$ and the full subcategory of $\operatorname{prin}(kI)$ formed by the modules X such that $\mathbf{csupp}\, X \subseteq J$.

LEMMA. (*a*) *If X in* $\operatorname{prin}(kI)$ *is indecomposable and is such that* $\mathbf{csupp}\, X \nsubseteq J$, *then* $X \simeq (S_i)^{\wedge}$ *for some* $i \in I' \setminus J$ *or* $X \simeq (S_j)^{\vee}$ *for some* $j \in I'' \setminus J$. *In particular, I is of finite prinjective type if and only if J is so.*

(*b*) *The quadratic form q^I is weakly positive if and only if q^J is so.*

Proof. (a) A direct summand $e_i A$ of X' or DBe_i of X'' with $i \in I \setminus J$ gives rise to a direct summand of $X = (X', X'', \varphi)$ isomorphic to $(e_i A, 0, 0)$ or $(0, DBe_i, 0)$, respectively.

(b) The condition is necessary, since $q^J = q^I \eta$, where $\eta : \mathbb{Z}^J \to \mathbb{Z}^I$ denotes the canonical injection. The condition is sufficient, since we have $q^I(x) \geq q^J \pi(x)$ for all $x \in \mathbb{N}^I$, where $\pi : \mathbb{Z}^I \to \mathbb{Z}^J$ denotes the canonical projection. ∎

(3.2) In [7] Bongartz proved that the Auslander–Reiten quiver Γ_Λ of a finite-dimensional, Schurian, directed k-algebra Λ has a preprojective component if the algebra Λ satisfies the separation condition [6]. We will follow his lines to show the existence of a preprojective component in $\Gamma(\operatorname{prin}(kI))$, but we shall use a different notion of "separation" which is more suited to our situation; it is based on the notion of connectedness of a bipartite poset given in 2.5.

For this purpose we fix a point $a \in I$ and consider the proper bipartite subposet of I defined as follows:

$$I^a = \begin{cases} I'_{\ngeq a} \cup I'' & \text{if } a \in I', \\ I' \cup I''_{\nleq a} & \text{if } a \in I''. \end{cases}$$

If B_a is the kI-module defined in (2.4) and Z in $\operatorname{prin}(kI)$ is an indecomposable direct summand of $\rho(B_a)$, then the coordinate support $\mathbf{csupp}\, Z$ is contained in I^a (2.5); we denote by $I^a(Z)$ the connected component of I^a (in the sense of (2.5)) which contains $\mathbf{csupp}\, Z$.

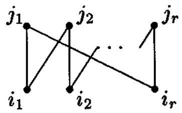
DEFINITION. (a) The bipartite poset $I = I' \cup I''$ is said to be ρ-separating at a point $a \in I$ if $I^a(Z_1) \neq I^a(Z_2)$ for any pair Z_1, Z_2 of nonisomorphic indecomposable direct summands of $\rho(B_a)$, where B_a is the kI-module defined in (2.4).

(b) The bipartite incidence k-algebra (1.1)

$$kI = \begin{pmatrix} kI' & M \\ 0 & kI'' \end{pmatrix}$$

of I is said to have a ρ-separation property if the bipartite poset $I = I' \cup I''$ is ρ-separating at every point $a \in I$.

DEFINITION. We say that the bipartite poset $I = I' \cup I''$ is $\tilde{\mathbb{A}}$-free (or crown-free) if it has no bipartite subposet of the form



$$, \quad r \geq 2,$$

(called a crown by J. A. de la Peña) with the bipartition $\{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_r\}$.

The following simple lemma will be essential in our future considerations.

LEMMA. (*a*) *A faithful bipartite poset $I = I' \cup I''$ is ρ-separating at every point $a \in I''$.*

(*b*) *A faithful $\tilde{\mathbb{A}}$-free bipartite poset $I = I' \cup I''$ is ρ-separating at every point $a \in I$.*

Proof. The statement (a) follows from Lemma 2.5 (b). The proof of (b) is simple and we leave it to the reader. ∎

(3.3) Recall from [18] that the category prin(kI) has almost split sequences. A connected component $\mathscr{C}$ of the translation quiver $\Gamma(\text{prin}(kI))$ (see [4], [10], [23]) is called *preprojective* if $\mathscr{C}$ has no oriented cycles and every X in $\mathscr{C}$ has the form $\Delta^{-r}B_i$ for some $r \in \mathbb{N}$ and $i \in I$, where $\Delta^- = \Delta^-_{kI}$ denotes the inverse of the Auslander–Reiten translation $\Delta = \Delta_{kI}$ in prin(kI).

PROPOSITION. *Let $I = I' \cup I''$ be a faithful, $\tilde{\mathbb{A}}$-free bipartite poset, $a \in I$ a vertex, B_a the kI-module defined in* (2.4), *Z an indecomposable direct summand of $\rho(B_a)$, and $J := I^a(Z)$. Suppose that $\Gamma(\text{prin}(kJ))$ has a preprojective component $\mathscr{C}$, and Y is a vertex of $\mathscr{C}$ such that Z is not a predecessor of Y. Then the following hold*:

(*a*) $\textbf{csupp}\, X \subseteq J$ *for every predecessor X of Y in* $\Gamma(\text{prin}(kI))$.

(*b*) $\Delta^-_{kI} Y \simeq \Delta^-_{kJ} Y$.

We remind the reader of the convention of considering $\operatorname{prin}(kJ)$ as a full subcategory of $\operatorname{prin}(kI)$ via the induction functor (2.2).

Proof. It is enough to show the following. Let X be an indecomposable module in $\operatorname{prin}(kI)$ such that there is an irreducible map $Y \to X$ or $X \to Y$ in $\operatorname{prin}(kI)$; then $\mathbf{csupp}\, X \subseteq J$.

Case 1. There is an irreducible map $Y \to X$ in $\operatorname{prin}(kI)$ and X is prin-projective.

First assume that $X \simeq B_b = (e_b kI)^\vee$ for some $b \in I'$, that is, Y is isomorphic to a direct summand of $(\operatorname{rad} e_b kI)^{\vee\wedge}$. Since $b \prec c$ for some $c \in (\mathbf{csupp} Y)'' \subseteq J''$, it is enough to prove that $b \in I^a$. (This would imply that $b \in J$, and therefore $\mathbf{csupp} B_b \subseteq J$.) Assume that $b \notin I^a$, that is, $a \in I'$ and $b \preceq a$. Since $Y \not\simeq Z$ and I is ρ-separating at a, we even have $b \prec a$. Choose a direct summand S of $\rho(B_b)$ such that $d \preceq a$ for some $d \in \mathbf{csupp} S$. We then have $J \subseteq I^b(S)$, since $(\mathbf{csupp} Z)'' \subseteq (\mathbf{csupp} S)''$, and the ρ-separation condition at b implies $Y \simeq S$, and therefore $d \in \mathbf{csupp} Y \subseteq I^a$, a contradiction.

Now assume that $X \simeq B_b = (S_b)^\vee$ for some $b \in I''$, that is, $Y \simeq (\tau_{kI} S_b)^{\vee\wedge}$. First note that $b \in I^a$. (Otherwise, since $\min(I''_{\succ b}) = (\mathbf{csupp} Y)'' \subseteq J \subseteq I^a$, we had $b = a$ and therefore $Y \simeq Z$, a contradiction.) We claim that $i \prec b$ for some $i \in (I^a)'$. (Once we have proved this, it follows that $b \in J$, since $i \prec c$ for any $c \in (\mathbf{csupp} Y)'' \subseteq J$.) To prove the claim we assume the contrary, that is, $a \in I'$ and $I'_{\prec b} \subseteq I'_{\preceq a}$. Let $(\mathbf{csupp} Y)'' = \{c_1, \dots, c_r\}$. If $r > 1$, we have $\varnothing \neq \min(I'_{\prec b}) \subseteq (\mathbf{csupp} Y)'$ (Lemma 2.5 (b)), and therefore $(\mathbf{csupp} Y)' \cap I'_{\preceq a} \neq \varnothing$, a contradiction. Thus $r = 1$, and $Y \simeq D(kJe_{c_1})^\wedge \in \operatorname{prin}(kJ)$ is prin-injective. Now choose a sequence of minimal length in J of the form $d_0 \succ d_1 \prec d_2 \succ \cdots \prec d_m$, where $d_0 \in \mathbf{csupp} Z$, $d_m \succeq c_1$, and $d_i \in J''$ iff i is even. ($m = 0$ is possible, but m is necessarily even.) The bipartite subposet of J formed by the minimal and the maximal elements of $\{d_1, \dots, d_m\} \cup \mathbf{csupp} Z$ is connected and gives rise to an indecomposable module W is $\operatorname{prin}(kJ)$ (2.5). Moreover, there are nonzero homomorphisms,

$$Z \to W \to D\big(kJe_{d_m}\big)^\wedge \to D\big(kJe_{c_1}\big)^\wedge \simeq Y,$$

which yields a contradiction.

To treat the remaining cases we need the following.

LEMMA. *If $i \in J''$ and $j \in I''$ are such that $i \prec j$, then $j \in J''$.*

Proof. If $i \notin \mathbf{csupp} Z$, there is some $\kappa \in J'$ such that $\kappa \prec i$, and therefore $\kappa \prec j$. Since $i \in I^a$ implies $j \in I^a$, we conclude that $j \in J$. The same argument applies if $i \in \mathbf{csupp} Z$ and $a \in I''$. In the remaining case, i belongs to $\max I''$, and there is nothing else to show. ∎

Case 2. There is an irreducible map $X \to Y$ in $\operatorname{prin}(kI)$, and Y is prin-projective.

First assume that $Y \simeq (e_b kJ)^{\vee} \in \operatorname{prin}(kJ)$ for some $b \in J'$. By the lemma, Y is even prin-projective in $\operatorname{prin}(kI)$; this implies that $(\mathbf{csupp}\, X)'' \subseteq (\mathbf{csupp}\, Y)'' \subseteq J$. On the other hand, we have $(\mathbf{csupp}\, X)' = \min(I'_{\succ b}) \subseteq I^a$ (since $b \in J \subseteq I^a$), and therefore $\mathbf{csupp}\, X \subseteq J$.

Now assume that $Y \simeq (S_b)^{\vee} \in \operatorname{prin}(kJ)$ for some $b \in J''$. Since $(\mathbf{csupp}\, X)'' \subseteq J$ (lemma!), it is enough to show that $(\mathbf{csupp}\, X)' \subseteq I^a$. This is clear if $a \in I''$; so we may assume that $a \in I'$.

Let $(\mathbf{csupp}\, X)'' = \min(J''_{\succ b}) = \{c_1, \dots, c_r\}$. We claim that $r = 1$. Denote by V the faithful kJ-module with $Ve_i \simeq k$ for every $i \in J$. Let $V_1 \subseteq V$ be the submodule generated by $Ve_i, i \in (\mathbf{csupp}\, Z)'$, and V_2 the factor module of V cogenerated by Ve_i, $i \in (\mathbf{csupp}\, X)''$. If $r > 1$ we have $0 \neq \dim_k \operatorname{Ext}^1_{kJ}(S_b, V_2) \leq \dim_k \operatorname{Hom}_{kJ}(V_2, \tau_{kJ} S_b)$, and therefore there are nonzero morphisms between indecomposable kJ-modules.

$$V_1 \to V \to V_2 \to \tau_{kJ} S_b.$$

Applying $(-)^{\vee \wedge}$ (with respect to $\operatorname{prin}(kJ)$), we obtain the induced sequence of nonzero morphisms between indecomposables in $\operatorname{prin}(kJ)$. Since $(V_1)^{\vee\wedge} \simeq Z$ and $(\tau_{kJ} S_b)^{\vee\wedge} \simeq \rho_{kJ}(Y)$, we get a contradiction.

If $i \prec b$ for some $i \in (\mathbf{csupp}\, Z)'$, we have $I_{\preceq a} \subseteq I_{\preceq i} \subseteq I_{\preceq b}$. On the other hand, $I_{\preceq b}$ does not intersect $\mathbf{csupp}\, X = \mathbf{csupp}(\tau_{kI} S_b)^{\vee\wedge}$, since $r = 1$. Hence $\mathbf{csupp}\, X \subseteq I^a$, and we can assume further on that $i \not\prec b$ for all $i \in (\mathbf{csupp}\, Z)'$.

Choose a sequence of minimal length in J of the form $d_0 \succ d_1 \prec d_2 \succ \cdots \prec d_m$, where $d_0 \in \mathbf{csupp}\, Z$, $d_m \succeq c_1$, and $d_l \in J''$ iff l is even. We claim that $m > 0$ and $d_{m-1} \prec b$. Indeed; otherwise the bipartite subposet of J formed by the minimal and the maximal elements of $\{d_1, \dots, d_m\} \cup \mathbf{csupp}\, Z$ gives rise to an indecomposable module W in $\operatorname{prin}(kJ)$ (2.5) and to a sequence of nonzero morphisms as follows, a contradiction:

$$Z \to W \to (S_b)^{\vee} \simeq Y.$$

Assume now that $(\mathbf{csupp}\, X)' \not\subseteq I^a$, that is, $i \preceq a$ for some $i \in (\mathbf{csupp}\, X)'$. Consider the chain $i \preceq a \prec d_0 \succ d_1 \prec \cdots \succ d_{m-1} \prec d_m \succeq c_1 \succ i$. Since I is $\tilde{\mathbb{A}}$-free, we have $i \prec d_l$ for all l, in particular, $i \prec d_{m-1} \prec b$. Using Lemma 2.5 (b) and $r = 1$, we obtain $i \notin \mathbf{csupp}\, X$, a contradiction.

The general case: We proceed by induction on the number $\nu(Y)$ of predecessors of Y in $\mathscr{C}$. If $\nu(Y) = 1$, then the module Y in $\operatorname{prin}(kJ)$ is prin-projective of the form $Y = (S_c)^{\vee}$ for some $c \in \max J' \subseteq \max I'$ (lemma!). In particular, Y is a source in $\Gamma(\operatorname{prin}(kI))$ and Case 1 applies.

In the induction step, we consider first the case in which there exists an irreducible morphism $X \to Y$. We may assume that Y is not prin-projective. Since $\nu(\Delta_{kJ}Y) < \nu(Y)$, induction implies $\Delta_{kI}^{-}\Delta_{kJ}Y \simeq \Delta_{kJ}^{-}\Delta_{kJ}Y \simeq Y$, and therefore $\Delta_{kJ}Y \simeq \Delta_{kI}Y$, which yields $\mathbf{csupp}\, X \subset J$. Finally, if there exists an irreducible morphism $Y \to X$, we may assume that X is not prin-projective. Then $\mathbf{csupp}\,\Delta_{kI}X \subset J$, as proved above. Since $\nu(\Delta_{kI}X) < \nu(Y)$, induction implies that $X \simeq \Delta_{kI}^{-}\Delta_{kI}X \simeq \Delta_{kJ}^{-}\Delta_{kI}X \in \operatorname{prin}(kJ)$. ▮

(3.4) THEOREM. *If I is a faithful $\tilde{\mathbb{A}}$-free bipartite poset, then the Auslander-Reiten quiver $\Gamma(\operatorname{prin}(kI))$ of $\operatorname{prin}(kJ)$ has a preprojective component.*

Proof. By induction on the cardinality of I. Follow the proof of Theorem 2.5 of [7], replacing Λ and Γ_Λ with kI and $\Gamma(\operatorname{prin}(kI))$, respectively; $R_i(a)$ and $B_i(a)$ by a direct summand Z of $\rho(B_a)$ and $k(I^a(Z))$ respectively; S by B_a, where $a \in \max I$; and τ^- by Δ^-; and using Proposition 3.2 instead of [7, Lemma 2.4 (b)]. Note that $I^a(Z)$ is faithful unless its cardinality is less than or equal to 1. ▮

(3.5) COROLLARY. *For any bipartite poset I of finite prinjective type, the following holds*:

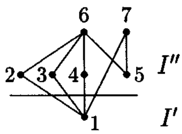
(*a*) *Each connected component of the Auslander–Reiten quiver* $\Gamma(\operatorname{prin}(kI))$ *is preprojective*.

(*b*) *The map $X \mapsto \mathbf{cdn}\, X$ induces a bijection between the isomorphism classes of indecomposable objects in* $\operatorname{prin}(kI)$ *and the positive roots of q^I, that is, the vectors $v \in \mathbb{N}^I$ such that $q^I(x) = 1$.*

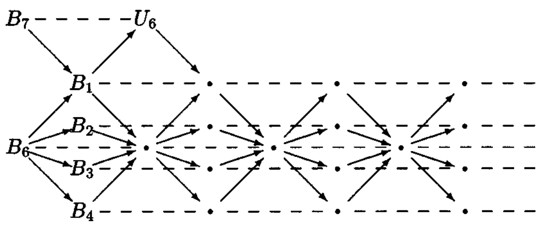
Proof. (a) The faithful part J of I is of finite prinjective type and $\tilde{\mathbb{A}}$-free. Therefore, by (3.4), each component of $\Gamma(\operatorname{prin}(kJ))$ is preprojective. The vertices of $\Gamma(\operatorname{prin}(kI))$ which do not belong to $\Gamma(\operatorname{prin}(kJ))$ are given by the prin-projective, prin-injective modules $S_i^{\wedge}$, $i \in I' \setminus J$, and $S_j^{\vee}$, $j \in I'' \setminus J$ (3.1). Now an irreducible map in $\operatorname{prin}(kI)$ ending at $S_i^{\wedge}$, $i \in I' \setminus J$, starts at $S_j^{\wedge}$ for some $j \in \min(I'_{\succ i}) \subseteq I' \setminus J$ (Lemma 2.5 (a)), and a similar statement holds for the irreducible maps starting in $S_j^{\vee}$, $j \in I'' \setminus J$. It follows that $\Gamma(\operatorname{prin}(kI))$ has no oriented cycle, and therefore each component is preprojective.

(b) follows from (a) and [18, 4.13]. ▮

Remark. If $I = I' \cup I''$ is a bipartite poset of infinite prinjective type and J denotes the faithful part of I, then $\Gamma(\operatorname{prin}(kI))$ does not necessarily have a preprojective component if $\Gamma(\operatorname{prin}(kJ))$ has one. For instance, consider the following bipartite poset I:



The faithful part of I is $J = I \setminus \{5\}$, and $\Gamma(\mathrm{prin}(kJ))$ has a preprojective component of the following form:



In particular, the prinjective module U_7 has infinitely many predecessors in $\Gamma(\mathrm{prin}(kJ))$ and therefore in $\Gamma(\mathrm{prin}(kI))$. Since by Lemma 2.5 (c) there are irreducible maps $U_6 \to B_5 \leftarrow U_7$, $\Gamma(\mathrm{prin}(kI))$ has no preprojective component. ∎

(3.6) Here we prove the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) in Theorem 1.2.

Proof. (i) $\Leftrightarrow$ (ii) By Lemma 3.1, we can assume that the bipartite poset I is faithful. We can further assume that I is $\tilde{\mathbb{A}}$-free, since this is implied by each of the conditions (i) and (ii). By Theorem 3.4, the Auslander−Reiten quiver $\Gamma(\mathrm{prin}(kI))$ has a preprojective component, and the equivalence of (i) and (ii) follows from [18, Proposition 4.13]. One obtains an alternative proof of the equivalence (i) $\Leftrightarrow$ (ii) by applying the arguments used in the proof of [23, Theorem 11.94].

(i) $\Rightarrow$ (iv) The existence of a preprojective component $\tilde{\mathscr{P}}(I)$ of $\Gamma(\mathrm{prin}(kI))$ is a consequence of Theorem 3.4. Since $\tilde{\mathscr{P}}(I)$ is finite, the equality $\Gamma(\mathrm{prin}(kI)) = \tilde{\mathscr{P}}(I)$ follows from a well-known criterion of M. Auslander (see [23, Theorem 11.44] and its proof).

(iv) ⇒ (i) We recall from [18] that $\operatorname{prin}(kI)$ has a finite complete set $E_1, \ldots, E_s$, $s = |I|$, of indecomposable relative injective modules. Since $\Gamma(\operatorname{prin}(kI)) = \tilde{\mathscr{P}}(I)$, the modules $E_1, \ldots, E_s$ are preprojective, and therefore the set of isomorphism classes of indecomposable modules X in $\operatorname{prin}(kI)$ such that $\operatorname{Hom}_{kI}(X, E_i) \neq 0$ is finite for every i. It follows that the category $\operatorname{prin}(kI)$ is of finite representation type.

(i) ⇔ (v) In view of the equivalences (i) ⇔ (ii) ⇔ (iv), the equivalence (i) ⇔ (v) follows by the arguments applied in the proof of [23, Theorem 11.94]. The details are left to the reader. ∎

4. CRITICAL BIPARTITE POSETS

In this section we prove the equivalence of the statements (ii) and (iii) in the main theorem (see Section 1).

(4.1) Recall from [11] that a *unit form* is an integral quadratic form q on $\mathbb{Z}^I$, where I is a nonempty finite set and $q(\varepsilon_i) = 1$ for every canonical basis vector $\varepsilon_i \in \mathbb{Z}^I$, $i \in I$. A unit form is *critical* if it is nonnegative of corank 1 and has a strictly positive radical generator. The $\mathbb{Z}$-equivalence classes of such critical unit forms correspond bijectively to the extended Dynkin diagrams $\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$, and $\tilde{\mathbb{E}}_8$; a complete classification is given in [11] (see also Zeldich [29]).

Critical unit forms play a decisive role in our context, because there is the following general criterion [11, 17]: a unit form q on $\mathbb{Z}^I$ is weakly positive if and only if $q(\varepsilon_i + \varepsilon_j) > 0$ for all $i, j \in I$ and q has no critical restriction. Here a *restriction* of q is a unit form of the form $q_J := qd_J$, where $J \subseteq I$ is a nonempty subset, and

$$d_J : \mathbb{Z}^J \to \mathbb{Z}^I$$

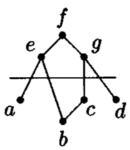
denotes the canonical embedding.

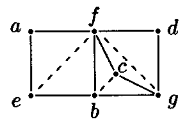
Now let $I = I' \cup I''$ be a bipartite poset and consider the associated Tits quadratic form $q^I : \mathbb{Z}^I \to \mathbb{Z}$ defined by the formula

$$q^I(x) = \sum_{i \in I} x_i^2 + \sum_{\substack{i \prec j \\ i, j \in I'}} x_i x_j + \sum_{\substack{i \prec j \\ i, j \in I''}} x_i x_j - \sum_{\substack{i \prec j \\ i \in I', j \in I''}} x_i x_j .$$

Clearly, q^I is a unit form with $q(\varepsilon_i + \varepsilon_j) > 0$ for all $i, j \in I$, and we have $(q^I)_J = q^J$ for every subset $J \subseteq I$ if we equip J with the full bipartite subposet structure of I. This shows that q^I is weakly positive if and only if I has no full bipartite subposet J where q^J is critical. So our task is to sift out from the set of all critical unit forms described in [11] those which are of the fom q^J for a bipartite poset J. The resulting list of bipartite posets will be that of Figure 1.

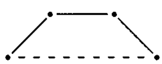
(4.2) Given a unit form q on $\mathbb{Z}^I$, we denote by $q_{ij} = q(\varepsilon_i + \varepsilon_j) - 2$ the coefficient of the term $x_i x_j$ in $q(x)$ for $i \neq j \in I$ and remind the reader of the convention of depicting q by a bigraph with vertex set I, $-q_{ij}$ solid edges between i and j if $q_{ij} < 0$, and q_{ij} dotted edges between i and j if $q_{ij} > 0$. For instance, the following figure shows a bipartite poset I together with the bigraph of q^I:

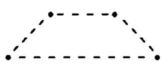
I 

q^I 

LEMMA. *The quadratic form q^I of a bipartite poset $I = I' \cup I''$ satisfies the following condition*:

(*) *The bigraph of q^I has neither multiple edges nor a full subbigraph isomorphic to any of the following bigraphs*:

p_1 

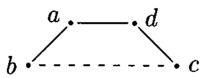
p_2 

p_3

p_4 

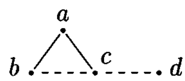
p_5 

Proof. First note that the transitivity of $\preceq$ implies that any restriction of q^I of the form •—•---• or •---•---• corresponds to elements $a, b, c \in I$ with a, c incomparable and $a \prec b \succ c$ or $a \succ b \prec c$. Now assume that q^I has a restriction of the form



or

and consider the triples a, b, c and b, c, d. We may assume that $a \in I'$ and obtain then $a \prec b \succ c \prec d$ with $b, c \in I''$ and $d \in I'$, which contradicts the bipartiteness. Similarly, a restriction of q^I of the form p_2 would give rise to a sequence $a_1 \prec a_2 \succ a_3 \prec a_4 \succ a_5 \prec a_1 \succ a_2$, a contradiction. Finally, assume that q^I has a restriction of the following form, where $a \in I'$:

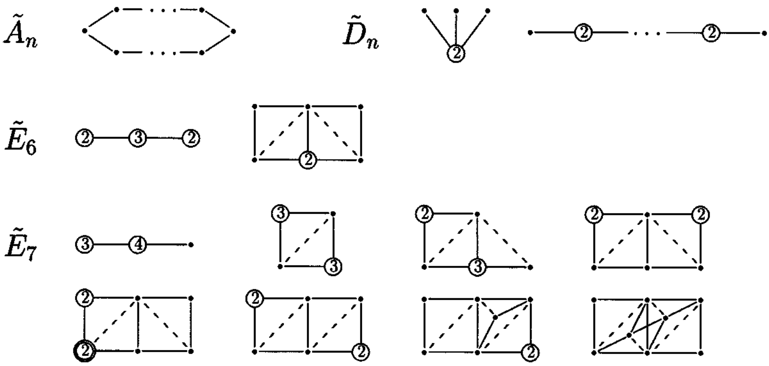


Applying the above observation to the triples a, c, d and b, c, d, we obtain $a \prec b \prec c$ with $b, c \in I''$, and any element of I which is comparable with b is also comparable with a or with c. This shows q^I has no restriction of the form p_4 or p_5. ∎

(4.3) Let us describe now the critical unit forms which satisfy condition $(*)$ of the above lemma. We will see that they coincide with the quadratic forms of the bipartite posets of Figure 1.

First note that the stretching procedure [11, 2.3] induces an equivalence relation on the set of critical unit forms such that all elements of a fixed equivalence class have a common restriction and can be obtained from it by branch extensions at certain vertices (see definition below). To be more precise, such an equivalence class is given by a unit form q', say on $\mathbb{Z}^J$, together with a subset $K \subseteq J$ and a tuple $(r_i)_{i \in K}$ of integers > 1, which indicate that, at every $i \in K$, a branch extension of q' of cardinality r_i has to be made. We depict such a triple $(q', K, (r_i))$ by the bigraph of q' with labels r_i at the vertices $i \in K$ and call it a *truncated* unit form of *type* Δ if the critical unit forms obtained from it are of (extended Dynkin) type Δ. For instance, the truncated unit forms of type $\tilde{\mathbb{E}}_n$ are those described by the truncated diagrams in Theorem 2.9 of [11] (the numbers attached there to the nonencircled vertices are not relevant here).

To describe all critical unit forms which satisfy (4.2)$(*)$, we have to examine which truncated unit forms satisfy (4.2)$(*)$ and which branch extensions of them do so. An inspection of the lists 2.1 and 2.9 of [11] shows that, for any of the types $\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{E}}_6$ and $\tilde{\mathbb{E}}_7$, the truncated unit forms satisfying (4.2)($*$) are those shown in the figure below



and that there are a further 33 of type $\tilde{\mathbb{E}}_8$, which we do not present here.

Next let us recall the notion of (quasi) branch [12, 2.1] of a unit form q on $\mathbb{Z}^I$. Denote by 0 an element not in I. A nonempty subset $A \subseteq I$ is a *branch* of q if there is a "nilpotent" map $\nu : A \cup \{0\} \to A \cup \{0\}$ (that is, $\nu(0) = 0$ and $\nu^r(A) = \{0\}$ for some $r \geq 1$) such that the following conditions are satisfied:

(a) For all $i, j \in A$,

$$q_{ij} = \begin{cases} -1 & \text{if } \nu(i) = j \text{ or } \nu(j) = i, \\ 1 & \text{if } \nu(i) = \nu(j), \\ 0 & \text{otherwise}. \end{cases}$$

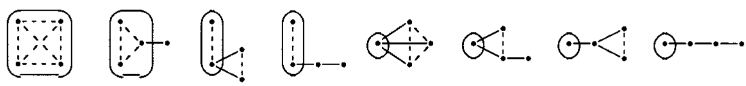
(b) For all $k \notin A$,

$$q_{ki} = \begin{cases} q_{kj} & \text{if } i, j \in \underline{A} := \{l \in A \mid \nu(l) = 0\}, \\ 0 & \text{if } i \in A \setminus \underline{A}. \end{cases}$$

The branches of cardinality r correspond bijectively to the (isomorphism classes) of nilpotent endomaps of the set $\{0, 1, \ldots, r\}$. The set $\underline{A}$ is called the set of *joint vertices* of the branch A. We say that q is obtained from a restriction q_J by *branch extensions* at the vertices $a_1, \ldots, a_s$ if there are pairwise disjoint branches $A_1, \ldots, A_s$ of q such that $a_i \in \underline{A_i}$ for $i = 1, \ldots, s$ and $J \cap (\bigcup_i A_i) = \{a_1, \ldots, a_s\}$.

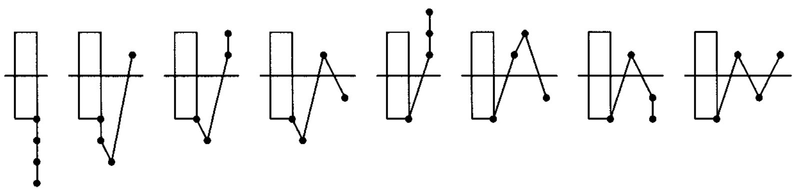
Now a branch A of q satisfies (4.2)($*$) if and only if q_A has no restriction of the form •—•---•—•, or, equivalently, the defining map ν of A satisfies $\operatorname{Im}\nu = \nu^{\mathbb{N}}(i)$ for some $i \in A$. There are 2^{r-1} branches of cardinality r satisfying this condition. For instance, for $r = 4$, there are the following eight branches, where the encircled parts denote the sets of joint vertices:

The critical unit forms which satisfy (4.2)($*$) can be described now as follows. Take any of the truncated unit forms q' described above and perform an extension by one of the 2^{r-1} branches described above at any vertex with label r unless the label is r. In this case, q' has a restriction



of the form r----•—•, so that a vertex with label r admits only an extension by the branch of cardinality r having no solid edge, that is, where all vertices are joint vertices.

One easily verifies that the quadratic forms of the bipartite posets of Figure 1 are the critical unit forms just described. The following figure illustrates for $r = 4$ how the 2^{r-1} possible extensions of a vertex with label r of a poset J correspond to the 2^{r-1} branch extensions at the corresponding vertex of q^J (compare with the figure above).



It remains to show that Figure 1 contains up to duality all critical bipartite posets. This rests on the remarkable fact that every bipartite poset of Figure 1 is uniquely determined up to isomorphism or duality by its quadratic form. This can be seen by using the argument given in the proof of Lemma 4.2 and is left to the reader.This, together with the results of (3.6), completes the proof of Theorem 1.2. ∎

Remark. By applying Theorem 1.2 to the bipartite posets $I = I' \cup I''$, with $I'' = \max I$, the set of all maximal elements of I and $I' = I \setminus \max I$, we get Theorem 3.1 in [22].

5. A MATRIX PROBLEM INTERPRETATION OF PRINJECTIVE MODULES OVER *kI*

We finish the paper by presenting a useful interpretation of the isomorphism classes of prinjective modules over the bipartite algebra (see (1.1)),

$$kI = \begin{pmatrix} kI' & M \\ 0 & kI'' \end{pmatrix},$$

in terms of orbits of partitioned matrices with respect to an algebraic group actions. This allows us to view the classification of indecomposable modules in $\text{prin}(kI)$ as a matrix problem in the sense of Drozd [8] (see also [10] and [23, Sections 1.1 and 17.9]). The interpretation is analogous to that one given in [22] for multipeak posets I, that is, finite posets I equipped with the bipartition $I = I' \cup I''$, where I'' consists of all maximal elements of I and $I' = I \setminus I''$.

(5.1) Let $I = I' \cup I''$ be a bipartite poset. Given a vector $v = (v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$, we consider the irreducible algebraic k-variety (in the Zariski topology),

$$\mathbf{prin}^{kI}_{(v',v'')} = \operatorname{Hom}_{kI}\big(P(v') \otimes_{kI'} M_{kI''}, Q(v'')\big),$$

and the natural algebraic group action,

$$\star : \mathbf{G}(v', v'') \times \mathbf{prin}^{kI}_{(v',v'')} \to \mathbf{prin}^{kI}_{(v',v'')},$$

where

$$\mathbf{G}(v', v'') = \operatorname{Aut}_{kI'}(P(v')) \times \operatorname{Aut}_{kI''}(Q(v'')),$$

$P(v') = \oplus_{j \in I'} (e_j kI')^{v'(j)}$, and $Q(v'') = \oplus_{p \in I''} D(kI'' e_p)^{v''(p)}$ (see [22], [13], [27]). It is clear that $\mathbf{G}(v', v'')$ is an affine algebraic group acting on $\mathbf{prin}^{kI}_{(v',v'')}$ in a natural way.

Note that there is a bijection between the isomorphism classes of modules X in $\text{prin}(kI)$ with $\mathbf{cdn}(X) = (v', v'')$ and the $\mathbf{G}(v', v'')$-orbits in $\mathbf{prin}^{kI}_{(v',v'')}$, given by attaching to the prinjective module $X = (X', X'', \varphi)$

the composed kI''-homomorphism

$$f_X = \left[P(v') \otimes_{kI'} M \simeq X' \otimes_{kI'} M \xrightarrow{\varphi} X'' \simeq Q(v'') \right]$$

(apply the arguments in [23, Section 15.2], [22] and [13]).

(5.2) We note that given $p \in I''$, the injective envelope $D(kI'' e_p))$ of top $e_p kI''$ (viewed as a k-linear representation of the poset I) is the constant diagram having the space k over all $j \preceq p$ and the space zero elsewhere. It follows that

$$\mathrm{Hom}_{kI}\big(e_i kI' \otimes_A M_B, D\big(kI'' e_p\big)\big) \simeq \begin{cases} k & \text{for all } i \preceq p, \\ 0 & \text{for all } i \not\preceq p. \end{cases}$$

(5.3) Assume that $I = I' \cup I''$ and $I' = \{1, \dots, n\}$, $I'' = \{n+1, \dots, n + m\}$. Without loss of generality, we can suppose that the partial order $\prec$ in I is such that $i \prec j$ implies that $i < j$ in the natural order.

Following [23, Section 15.2], [22], [13], and applying (5.2), one shows that the correspondence $X \mapsto f_X$ allows us to identify the algebraic k-variety $\mathbf{prin}^{kI}_{(v', v'')}$ with the affine k-variety $\mathbf{Mat}^{I}_{(v', v'')}$ of all partitioned matrices of the form (compare with Chapter 2 of [23])

$$A = \begin{array}{c} \left[\begin{array}{c|c|c|c} A_{1n+1} & A_{2n+1} & \cdots & A_{nn+1} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline A_{1n+m} & A_{2n+m} & \cdots & A_{nn+m} \end{array}\right] \begin{array}{l} \}v''(n+1) \\ \quad\vdots \\ \}v''(n+m) \end{array} \\ \underbrace{\qquad}_{v'(1)} \quad \underbrace{\qquad}_{v'(2)} \quad \cdots \quad \underbrace{\qquad}_{v'(n)} \end{array}$$

with coefficients in k, where $A_{in+t} = 0$ if $i \not\preceq n + t, t = 1, \dots, m$.

The affine algebraic group $\mathbf{G}(v', v'')$ is isomorphic to the group $\mathbf{G}^{I}_{(v', v'')}$ generated by the following (I', I'')-elementary transformations on matrices in $\mathbf{Mat}^{I}_{(v', v'')}$

($\mathbf{E_1}$) All $(I'')^{op}$-elementary transformations on rows from the jth row block to the ith row block for all $i \preceq j \in I''$

($\mathbf{E_2}$) All I'-elementary transformations on columns from the ith column block to the jth column block for all $i \preceq j$ in I'

Here, by an elementary transformation on a row (resp. a column) ω of A, we mean the addition to ω a row (resp. a column) $\omega' \neq \omega$ of A multiplied by a nonzero scalar $\lambda \in k$.

The action (5.1) corresponds to the obvious algebraic group action

$$\bullet : \mathbf{G}^{I}_{(v', v'')} \times \mathbf{Mat}^{I}_{(v', v'')} \to \mathbf{Mat}^{I}_{v', v'')}.$$

Note that if $\rho \in \mathbf{G}^I_{(v',v'')}$ is an elementary transformation, $A \in \mathbf{Mat}^I_{(v',v'')}$ is a partitioned matrix and

$$\rho \cdot A = \begin{array}{c} \left.\begin{array}{|c|c|c|c|} \hline (\rho \cdot A)_{1n+1} & (\rho \cdot A)_{2n+1} & \cdots & (\rho \cdot A)_{nn+1} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline (\rho \cdot A)_{1n+m} & (\rho \cdot A)_{2n+m} & \cdots & (\rho \cdot A)_{nn+m} \\ \hline \end{array}\right. \begin{array}{l} \}v''(n+1) \\ \vdots \\ \}v''(n+m) \end{array} \\ \underbrace{\qquad\qquad}_{v'(1)} \quad \underbrace{\qquad\qquad}_{v'(2)} \quad \cdots \quad \underbrace{\qquad\qquad}_{v'(n)} \end{array}$$

is the usual partitioned matrix obtained from A by applying the transformation ρ; then the partitioned matrix

$$\rho \bullet A = \begin{array}{c} \left.\begin{array}{|c|c|c|c|} \hline (\rho \bullet A)_{1n+1} & (\rho \bullet A)_{2n+1} & \cdots & (\rho \bullet A)_{nn+1} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline (\rho \bullet A)_{1n+m} & (\rho \bullet A)_{2n+m} & \cdots & (\rho \bullet A)_{nn+m} \\ \hline \end{array}\right. \begin{array}{l} \}v''(n+1) \\ \vdots \\ \}v''(n+m) \end{array} \\ \underbrace{\qquad\qquad}_{v'(1)} \quad \underbrace{\qquad\qquad}_{v'(2)} \quad \cdots \quad \underbrace{\qquad\qquad}_{v'(n)} \end{array}$$

in $\mathbf{Mat}^I_{(v',v'')}$ is defined by the formula

$$(\rho \bullet A)_{ij} = \begin{cases} (\rho \cdot A)_{ij} & \text{for } i \preceq j, \\ 0 & \text{for } i \not\preceq j. \end{cases}$$

From the discussion above it is not difficult to get the following result (see also [13, Proposition 2.9, Corollary 2.12]).

PROPOSITION. *Assume that k is a field, $I = I' \cup I''$ is a bipartite poset, and $I' = \{1, \ldots, n\}$, $I'' = \{n+1, \ldots, n+m\}$. Suppose that the partial order $\prec$ in I is such that $i \prec j$ implies that $i < j$ in the natural order. In the notation introduced above, the following hold:*

(*a*) *For any coordinate vector $v = (v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$ there exists a natural algebraic group isomorphism*

$$\lambda_{(v',v'')} : \mathbf{G}^I_{(v',v'')} \to H^{I''}_{v'} \times H^{I''}_{v''}$$

where $H_{v'}^{I'}$ *and* $H_{v''}^{I''}$ *are the groups consisting of all matrices of the following form*:

$$h' = \begin{bmatrix} h'_{11} & h'_{12} & \cdots & h'_{1n} \\ 0 & h'_{22} & \cdots & h'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h'_{nn} \end{bmatrix} \in \mathrm{Gl}(v'(1) + \cdots + v'(n), k),$$

and

$$h'' = \begin{bmatrix} h''_{1n+1} & h''_{1n+2} & \cdots & h''_{1n+m} \\ 0 & h''_{2n+2} & \cdots & h''_{2n+m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h''_{nn+m} \end{bmatrix}$$

$$\in \mathrm{Gl}(v''(n+1) + \cdots + v''(n+m), k),$$

respectively, and $h'_{ii} \in \mathrm{Gl}(v'(i), k)$, $h''_{jj} \in \mathrm{Gl}(v''(j), k)$, $h'_{ij} = 0$ *if* $i \not\preceq j$ *holds in* I', $h''_{ij} = 0$ *if* $i \not\preceq j$ *holds in* I''.

(*b*) *The action* $\bullet : \mathbf{G}^I_{(v',v'')} \times \mathbf{Mat}^I_{(v',v'')} \to \mathbf{Mat}^I_{(v',v'')}$ *corresponds via the isomorphism* $\lambda_{(v',v'')}$ *to the algebraic group action*

$$\bullet : \left(H_{v'}^{I'} \times H_{v''}^{I''}\right) \times \mathbf{Mat}^I_{(v',v'')} \to \mathbf{Mat}^I_{(v',v'')},$$

defined by the formula $(h', h'')\bullet A = h'' \circ A \circ (h')^{-1}$, *where* $A \circ h' = A'$ *and* $h'' \circ B = B'$ *are partitioned matrices of the form shown above, with* $A'_{sp} = (Ah')_{sp}$ *for* $s \preceq p$ *and* $a'_{sp} = 0$ *for* $s \not\preceq p$, $B'_{ij} = (h''B)_{ij}$ *for* $i \preceq j$, *and* $B'_{ij} = 0$ *for* $i \not\preceq j$.

(*c*) *For any coordinate vector* $v = (v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$, *there exist algebraic group isomorphisms* $\mathbf{G}(v) \overset{\simeq}{\to} \mathbf{G}^I_{(v',v'')} \overset{\lambda_{(v',v'')}}{\to} H_{v'}^{I'} \times H_{v''}^{I''}$ *and a* k-*variety isomorphism* $\mathbf{prin}^{kI}_{(v',v'')} \overset{\simeq}{\to} \mathbf{Mat}^I_{(v',v'')}$ *such that the following diagram*:

$$\begin{array}{ccc} \mathbf{G}(v', v'') \times \mathbf{prin}^{kI}_{(v',v'')} & \overset{*}{\longrightarrow} & \mathbf{prin}^{kI}_{(v',v'')} \\ \downarrow\simeq & & \downarrow\simeq \\ (H_{v'}^{I'} \times H_{v''}^{I''}) \times \mathbf{Mat}^I_{(v',v'')} & \overset{\bullet}{\longrightarrow} & \mathbf{Mat}^I_{(v',v'')} \end{array}$$

is commutative.

Following [22], we say that the bipartite algebra

$$kI = \begin{pmatrix} kI' & M \\ 0 & kI'' \end{pmatrix}$$

is of *tame prinjective type* if the full subcategory $\mathrm{prin}(kI)$ of $\mathrm{mod}(kI)$ is of tame representation type (see [23, Section 14.4]).

It follows from [18, Proposition 1.1] that the well-known theorem of Drozd [9] applies to $\mathrm{prin}(kI)$, and therefore the tame–wild dichotomy holds for the category $\mathrm{prin}(kI)$ (see also [27]).

COROLLARY. *Let $I = I' \cup I''$ be a bipartite poset and let k be an algebraically closed field.*

(a) *The following three conditions are equivalent*:

(a1) *The bipartite poset I is of finite prinjective type.*

(a2) *For every vector $(v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$, the number of $\mathbf{G}^I_{(v',v'')}$-orbits in the variety $\mathbf{Mat}^I_{(v',v'')}$ is finite.*

(a3) $\dim \mathbf{G}^I_{(v',v'')} > \dim \mathbf{Mat}^I_{(v',v'')}$ *for any vector $(v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$, where* $\dim$ *means the k-variety dimension.*

(b) *The bipartite algebra kI is of tame prinjective type if and only if for every coordinate vector $(v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$ there exists a constructible subset $\mathscr{C}(v', v'')$ of the subvariety $\mathbf{ind\,Mat}^I_{(v',v'')}$ of $\mathbf{Mat}^I_{(v',v'')}$ defined by the indecomposable $\mathbf{G}^I_{(v',v'')}$-orbits such that $\mathbf{G}^I_{(v',v'')} \bullet \mathscr{C}(v', v'') = \mathbf{ind\,Mat}^I_{(v',v'')}$ and* $\dim \mathscr{C}(v', v'') \leq 1$.

Proof. (a) It is easy to see that the equality

$$q^I(v', v'') = \dim G^I_{(v',v'')} - \dim \mathbf{Mat}^I_{(v',v'')}$$

holds for any vector $(v', v'') \in \mathbb{N}^{I'} \times \mathbb{N}^{I''}$. Then the equivalences (a1) $\Leftrightarrow$ (a2) $\Leftrightarrow$ (a3) easily follow by applying Theorem 1.2, Proposition 5.3, and the arguments used in the proof of Theorem 3.1 in [22]. The details are left to the reader.

(b) Apply the arguments given in the proof of Corollary 15.17 of [23], in the proof of Theorem 3.17 of [13] and in Theorem 3.1 of [14]. The details are left to the reader. ∎

Problem 1. Generalize Theorem 3.1 in [14] from two-peak thin posets of tame prinjective type to a class of bipartite posets.

Problem 2. Extend Theorem 2.1 in [16] from multipeak posets to arbitrary bipartite posets.

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